The Real Business Cycle Model

Weeks 07 & 08

Vivaldo Mendes
Instituto Universitário de Lisboa (ISCTE-IUL)
vivaldo.mendes@iscte-iul.pt

2026-04-20

1. What is the Real Business Cycle model?

The RBC model in a nutshell

- It is the Solow model, but …
- ***Remove growth*** because we are only interested in business cycle fluctuations.
  - So: $g_A = 0$, $g_L = 0$
- The savings rate (s) becomes ***endogenous***: households maximize intertemporal utility and savings wii adadpt to changes in the economy
- Add ***leisure*** to account for changes in hours of work
- Add ***shocks*** to technology (the A variable in the Solow model)
- Shocks lead to ***uncertainty*** in the economy
- To deal with uncertainty, agents formulate expectations according to the ***rational expectations*** hypothesis.

Major assumptions & a fundamental result

- There are ***no market failures***:
  - No asymmetric information
  - No public goods
  - No externalities
- There is ***no market power***:
  - Firms are price takers
  - Households are price takers
- Prices and wages are ***perfectly flexible***
- From those assumptions we can obtain a ***fundamental result***:
  - Money is completely neutral in the economy
  - Central banks are perfectly dispensable for improving welfare

How relevant is the RBC today?

- If we google the term "RBC model" we will find:

  - “A Google search for the exact phrase”Real Business Cycle model” returns approximately ***348,000 results***.”
- It is the workhorse model of modern macroeconomics:
  - It was the ***first DSGE model***
  - It was the first to use ***rational expectations***
  - It was the first to use the ***micro-foundations*** of modern macroeconomics
  - It was the first to use the ***computer*** to solve a modern macroeconomic model
- It has serious limitations:
  - It cannot explain the crucial importance of ***central banks***
  - It cannot explain why business cycles are a ***bad thing***
  - It cannot explain why ***unemployment*** is a bad thing
  - Some crucial assumptions look ***unrealistic***

2. Behavioral equations

Households

- Households ***maximize utility*** (u) over time
- Utility depends on ***consumption*** (c) and on ***hours worked*** (ℓ):

$$u(c_t, \ell_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \theta \frac{\ell_t^{1+\gamma}}{1+\gamma}$$

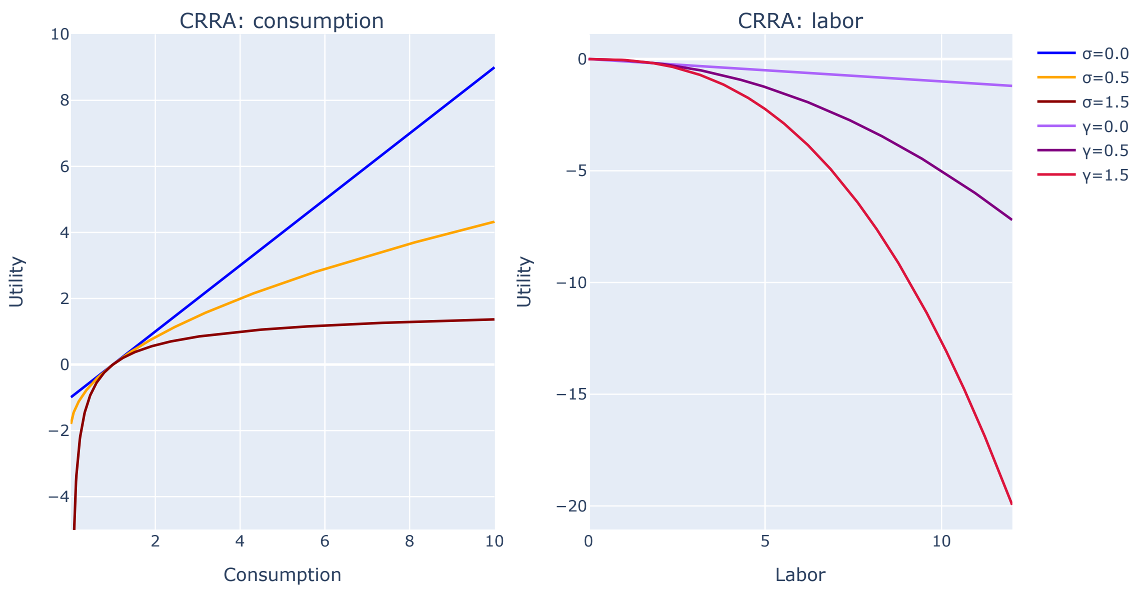
  - σ, γ : coefficients of relative risk aversion
  - θ : a scale parameter of the disutility of labor
- Intertemporal utility is ***discounted*** by a factor β, then

$$u(\cdot) = \overbrace{\beta^0 \cdot u(c_0, \ell_0)}^{\text{period } 0} + \overbrace{\beta^1 \cdot u(c_1, \ell_1)}^{\text{period } 1} + \overbrace{\beta^2 \cdot u(c_2, \ell_2)}^{\text{period } 2} + \ldots + \overbrace{\beta^T \cdot u(c_T, \ell_T)}^{\text{period } T}$$

- The ***discount factor*** is $\beta = 1/(1+d)$; while d is the subjective discount rate of future utility.

CRRA Utility Function

- The utility function: $u(c_t, \ell_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \theta \frac{\ell_t^{1+\gamma}}{1+\gamma}$



- Nonlinear with respect to consumption: $\sigma > 0$
- Linear in leisure (for simplicity): $\gamma = 0$

Firms

- Each firm ***produces goods & services*** with the following production function:

$$y_t = a_t k_t^{\alpha} \ell_t^{1-\alpha}$$

  - y is output per household; a is publicly available technology; k is capital per household; ℓ_t is the average hours worked per household; and α is the output/capital elasticity.

- Firms ***maximize profits*** and take wages and rental prices as given:

$$max\{\pi = y_t - w_t \ell_t - r_t k_t\}$$

  where w_t is the wage rate, r_t is the rental price of capital.

- As markets are ***fully competitive***, factor returns are equal to their marginal products:

$$w_t = \frac{\partial y_t}{\partial \ell_t} \quad \text{and} \quad r_t = \frac{\partial y_t}{\partial k_t}$$

Capital and labor accumulation

- Population remains constant over time. This means that the total ***number of households*** remains constant:

$$n_t = \overline{n}$$

- Therefore, as far as labor is concerned, only changes in the ***average hours worked*** per household (ℓ) can affect the level of output.
- ***Capital accumulation*** is given by the usual definition from national accounts:

$$k_{t+1} \equiv i_t + (1-\delta)k_t$$

- k represents capital per household, i investment per household, and δ the depreciation rate.

Technology

- Assume that technology (a_t) ***does not increase*** over time (there is no trend)
- a_t fluctuates around its steady state value $(\overline{a})$, due to ***exogenous shocks*** (ε_t)

$$\ln a_t = (1-\rho)\ln \overline{a} + \rho \ln a_{t-1} + \varepsilon_t \quad , \quad \rho < 1$$

- Why logarithms? To make things easier!
- Define:

$$\hat{a}_t = \ln a_t - \ln \overline{a}$$

- Then eq. (7) can be written as

$$\hat{a}_t = \rho \cdot \hat{a}_{t-1} + \varepsilon_t$$

  i.e., the log-deviation of technology from its steady state is an AR(1) process with $\rho < 1$, and mean=zero.

The macroeconomic constraint

- In a simple economy, ***output*** is either consumed by households (c_t) or invested in new capital (i_t):

$$y_t \equiv c_t + i_t$$

- We know that output is ***produced*** according to eq. (2):

$$y_t = a_t k_t^{\alpha} \ell_t^{1-\alpha}$$

- And ***capital*** accumulation is given by eq. (6):

$$k_{t+1} \equiv i_t + (1-\delta)k_t$$

- Inserting eq. (2′) & (6′) into (8) we get the overall ***macroeconomic constraint*** as:

$$a_t k_t^{\alpha} \ell_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta)k_t$$

A benevolent government

- The government maximizes the ***welfare*** of the representative household ***subject to*** the macroeconomic constraint.

$$\max_{\{c_t,\ell_t k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t,\ell_t) = \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \theta \cdot \ell_t \right)$$

$$\text{subject to:} \quad a_t k_t^{\alpha} \ell_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta) k_t$$

$$c_t, c_{t+1} \geq 0 \quad , \quad \ell_t, \ell_{t+1} \in (0,1) \quad , \quad 0 < \beta \leq 1$$

- The government knows that it needs also the other four equations:

$$k_{t+1} \equiv i_t + (1-\delta) k_t$$
$$y_t = a_t k_t^{\alpha} \ell_t^{1-\alpha}$$
$$y_t \equiv c_t + i_t$$
$$\ln a_t = (1-\rho) \ln \bar{a} + \rho \ln a_{t-1} + \varepsilon_t \quad , \quad \varepsilon_t \sim iid(0,\sigma^2)$$

---

3. Solving the intertemporal optimization of utility

The Lagrangian

- The intertemporal maximization of utility

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Big\{ \underbrace{u(c_t,\ell_t)}_{\text{utility}} + \lambda_t \underbrace{\left(a_t k_t^{\alpha} \ell_t^{1-\alpha} + (1-\delta) k_t - c_t - k_{t+1}\right)}_{\text{resource constraint}} \Big\}$$

  where λ_t stands for the ***Lagrangian multiplier***

- First Order Conditions (***FOCs***):
  - Write the Lagrangian for two consecutive periods $(t, t+1)$
  - Take first order conditions with respect to $c_t, k_{t+1}, \ell_t, \lambda_t$

  $\partial \mathcal{L} / \partial c_t = 0, \quad \partial \mathcal{L} / \partial k_{t+1} = 0, \quad \partial \mathcal{L} / \partial \ell_t = 0, \quad \partial \mathcal{L} / \partial \lambda_t = 0$

- It looks complicated, but it is not! What you need is ***PATIENCE!***

FOCs

- The Lagrangian function for periods t and $t+1$ is:

$$\mathcal{L} = \ldots + \beta^0 \{u(c_t,\ell_t) + \lambda_t (a_t k_t^{\alpha} \ell_t^{1-\alpha} + (1-\delta) k_t - c_t - k_{t+1})\} +$$
$$+ \beta^1 \{u(c_{t+1},\ell_{t+1}) + \lambda_{t+1} (a_{t+1} k_{t+1}^{\alpha} \ell_{t+1}^{1-\alpha} + (1-\delta) k_{t+1} - c_{t+1} - k_{t+2})\} + \ldots$$

- Now the FOCs, one by one (follow the colors for guidance):

$$\partial \mathcal{L}/\partial c_t = \beta^0 \left(u'_{c_t} - \lambda_t \right) = 0$$

$$\partial \mathcal{L}/\partial k_{t+1} = -\beta^0 \cdot \lambda_t + \beta^1 \cdot \lambda_{t+1} \left(\underbrace{\alpha \cdot a_{t+1} k_{t+1}^{\alpha-1} \ell_{t+1}^{1-\alpha} + 1 - \delta}_{\equiv r_{t+1}} \right) = 0$$

$$\partial \mathcal{L}/\partial \ell_t = \beta^0 \left[u'_{\ell_t} + \lambda_t (1-\alpha) \underbrace{a_t^{k_t^{\alpha} \ell_t^{-\alpha}}}_{=y_t/\ell_t} \right] = 0$$

$$\partial \mathcal{L}/\partial \lambda_t = \beta^0 \left(a_t k_t^{\alpha} \ell_t^{1-\alpha} + (1-\delta) k_t - c_t - k_{t+1} \right) = 0$$

FOCs, FOCs, and a definition

$$\partial \mathcal{L}/\partial c_t = \beta^0 \left(u'_{c_t} - \lambda_t \right) = 0$$

$$\partial \mathcal{L}/\partial k_{t+1} = -\beta^0 \lambda_t + \beta^1 \lambda_{t+1} \left(\underbrace{\alpha \cdot a_{t+1} k_{t+1}^{\alpha-1} \ell_{t+1}^{1-\alpha} + 1 - \delta}_{\equiv r_{t+1}} \right) = 0$$

$$\partial \mathcal{L}/\partial \ell_t = \beta^0 \left[u'_{\ell_t} + \lambda_t (1-\alpha) \underbrace{a_t k_t^{\alpha} \ell_t^{-\alpha}}_{=y_t/\ell_t} \right] = 0$$

$$\partial \mathcal{L}/\partial \lambda_t = \beta^0 \left(a_t k_t^{\alpha} \ell_t^{1-\alpha} + (1-\delta) k_t - c_t - k_{t+1} \right) = 0$$

- Notice that r_{t+1} is the value of one unit of capital at $t+1$, defined as:

$$r_{t+1} \equiv \alpha \cdot a_{t+1} k_{t+1}^{\alpha-1} \ell_{t+1}^{1-\alpha} + 1 - \delta = \alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta$$

Simplifying FOC1

- From FOC1, we know that:

$$\beta^0 \left(u'_{c_t} - \lambda_t \right) = 0 \Rightarrow \lambda_t = u'_{c_t}$$

- As from eq. (1), the ***marginal utility of consumption*** can be written as:

$$u'(c_t) = \frac{\partial u(c_t, \ell_t)}{\partial c_t} = c_t^{-\sigma}$$

- By inserting eq. (12) into (11), we get:

$$\lambda_t = c_t^{-\sigma}$$

- Obviously, if $\lambda_t = c_t^{-\sigma}$, then

$$\lambda_{t+1} = c_{t+1}^{-\sigma}$$

Simplifying FOC2

- From FOC2, we know that: $\beta^0 \lambda_t = \beta^1 \lambda_{t+1} r_{t+1}$.
- By inserting eq. (13) and (14) into this FOC2, we get:

$$c_t^{-\sigma} = \beta \cdot c_{t+1}^{-\sigma} \cdot r_{t+1}$$

- This is the famous ***Euler equation***. It gives the optimal level of consumption over time, which depends on the discount factor β, the risk aversion parameter σ and the net return on capital r_{t+1}.
- Notice that the Euler equation can be written as a ratio (ξ):

$$\xi \equiv \frac{c_{t+1}}{c_t} = \left(\beta \cdot r_{t+1}\right)^{1/\sigma}$$

Implying that: $\partial\xi/\partial r_{t+1} > 0 \quad , \quad \partial\xi/\partial\beta > 0 \quad , \quad \partial\xi/\partial\sigma < 0.$

Simplifying FOC3

- From FOC3, we know that:

$$\beta^0 \left[u'_{\ell_t} + \lambda_t (1-\alpha) \underbrace{a_t k_t^{\alpha} \ell_t^{-\alpha}}_{= y_t/\ell_t} \right] = 0$$

- As we know that from eq. (1) the ***marginal utility of working*** is given by:

$$u'(\ell_t) = \frac{\partial u(c_t, \ell_t)}{\partial \ell_t} = -\theta$$

- By inserting eq. (13) and (17) into (FOC3), we get:

$$-\theta + c_t^{-\sigma}(1-\alpha)\frac{y_t}{\ell_t} = 0 \quad \Rightarrow \quad \frac{y_t}{\ell_t} = \left(\frac{\theta}{1-\alpha}\right) c_t^{\sigma}$$

- This is the ***static equation*** that optimizes the labour supply.

Simplifying FOC4

- From FOC4, we know that:

$$\beta^0 \left(a_t k_t^{\alpha} \ell_t^{1-\alpha} + (1-\delta) k_t - c_t - k_{t+1} \right) = 0$$

- As $\beta^0 = 1$, this implies that the resource constraint is fully satisfied:

$$a_t k_t^{\alpha} \ell_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta) k_t$$

- No more simplifications are possible in this equation.

Simplified FOCs

- We have the 3 simplified FOCS:

$$
\begin{aligned}
c_t^{-\sigma} &= \beta \cdot c_{t+1}^{-\sigma} \cdot r_{t+1} \\
\frac{y_t}{\ell_t} &= \left(\frac{\theta}{1-\alpha}\right) c_t^{\sigma} \\
\underbrace{a_t k_t^{\alpha} \ell_t^{1-\alpha}}_{=y_t} &= c_t + \underbrace{k_{t+1} - (1-\delta) k_t}_{=i_t}
\end{aligned}
$$

- These involve 7 variables in total: $\{c_t, \ell_t, k_t, y_t, a_t, r_{t+1}, i_t\}$
- So we need to include more 4 equations to solve the model.
- See next slide for the remaining equations.

The remaining equations

$$
\begin{aligned}
r_{t+1} &= \alpha a_{t+1} k_{t+1}^{\alpha-1} \ell_{t+1}^{1-\alpha} + 1 - \delta \\
y_t &= a_t k_t^{\alpha} \ell_t^{1-\alpha} \\
\ln a_t &= (1-\rho) \ln \overline{a} + \rho \ln a_{t-1} + \varepsilon_t \\
y_t &= c_t + i_t
\end{aligned}
$$

A non-linear problem

The solution to the model involves 7 non-linear equations and 7 variables $\{c_t, \ell_t, k_t, y_t, a_t, r_{t+1}, i_t\}$:

$$
\begin{aligned}
c_t^{-\sigma} &= \beta \cdot c_{t+1}^{-\sigma} \cdot r_{t+1} \\
\frac{y_t}{\ell_t} &= \left(\frac{\theta}{1-\alpha}\right) c_t^{\sigma} \\
r_{t+1} &\equiv \alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \\
y_t &= a_t k_t^{\alpha} \ell_t^{1-\alpha} \\
k_{t+1} &\equiv (1-\delta) k_t + i_t \\
\ln a_t &= (1-\rho) \ln \overline{a} + \rho \ln a_{t-1} + \varepsilon_t \\
y_t &\equiv c_t + i_t
\end{aligned}
$$

The steady-state

- To compute the ***steady-state*** of a variable x, impose the usual condition:

$$
x_{t+1} = x_t = \overline{x}
$$

- Let us start with the Euler equation (eq. 15):

$$c_t^{-\sigma} = \beta(c_{t+1}^{-\sigma} \cdot r_{t+1}) \Rightarrow \frac{(\overline{c})^{-\sigma}}{(\overline{c})^{-\sigma}} = \beta \cdot \overline{r}$$
$$\overline{r} = \beta^{-1}$$

- Using the (VOC) definition $r_{t+1} \equiv \alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta$, and the result in eq. (22), we get:

$$\beta^{-1} \equiv \alpha\left(\frac{\bar{y}}{\bar{k}}\right) + 1 - \delta \quad \Rightarrow \frac{\bar{y}}{\bar{k}} = \frac{\beta^{-1} + \delta - 1}{\alpha}$$

The steady-state (cont.)

- From eq. (6′), we can obtain:

$$\overline{k} = (1-\delta)\overline{k} + \overline{\imath} \quad \Rightarrow \quad \frac{\bar{i}}{\bar{k}} = \delta$$

- From eq. (23) we know that $\frac{\bar{y}}{\bar{k}} = \frac{\beta^{-1}+\delta-1}{\alpha}$, and from (24) we have $\frac{\bar{i}}{\bar{k}} = \delta$. Therefore, we can obtain after some manipulations:

$$\frac{\bar{i}}{\bar{y}} = \frac{\frac{\bar{i}}{\bar{k}}}{\frac{\bar{y}}{\bar{k}}} = \phi \quad , \quad \text{with} \quad \phi \equiv \frac{\alpha\delta}{\beta^{-1} + \delta - 1}$$

The steady state (cont.)

- From eq.(8′) we have $y_t = c_t + i_t$. By dividing both sides by y_t, and knowing that from (25) we have $\frac{\bar{i}}{\bar{y}} = \phi$, we get:

$$\frac{\overline{c}}{\bar{y}} = 1 - \frac{\bar{i}}{\bar{y}} = 1 - \phi$$

- From eq. (18′), we can obtain:

$$\frac{\overline{y}}{\bar{\ell}} = \frac{\theta}{1-\alpha}(\overline{c})^{\sigma}$$

- Finally, by construction, without the shocks, we can obtain the following in the steady-state:

$$\ln \overline{a} = (1-\rho)\ln \overline{a} + \rho \ln \overline{a} \quad \Rightarrow \quad 0 = 0$$

4. Linearizing the model near the steady state

The problem of non-linearity

- The model is non-linear, and could be easily simulated by using a computer.

- However, there is a ***problem*** if we take into account the uncertainty that flows from the shocks that may hit the economy
- If private agents formulate ***expectations*** about the future states of the endogenous variables, how can do it if the model is non-linear?
  - We can formulate expectations of variables only if they are stationary: the expectations operator is linear!
- Therefore, the consideration of uncertaint and nonlinearity requires a trick: ***linearize the model in the neighborhood of the steady state***.

What is linearization?

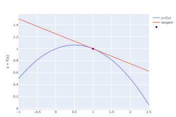
- Suppose we have a nonlinear function

$$y = f(x) = \frac{1}{4}(4 + x - x^2)$$

- We want to approximate it around a point $\bar{x} = 1$.
- We can approximate it by its tangent line at $\bar{x} = 1$, given by:

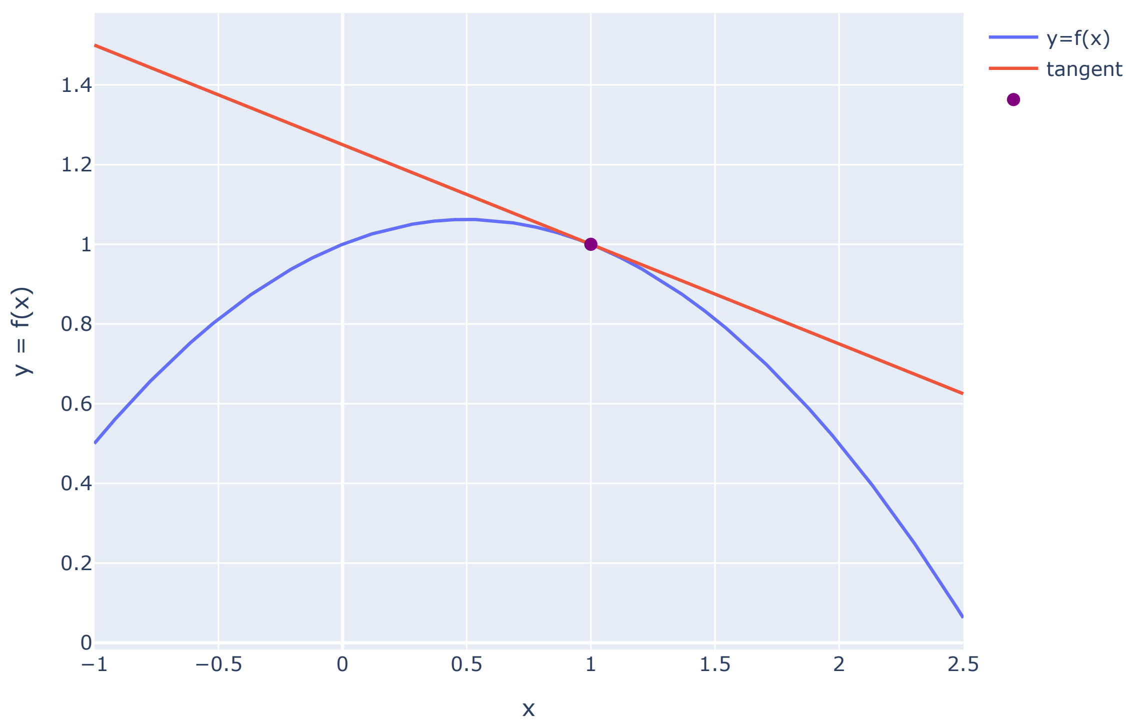
$$y = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = -1/4(x + 5/4)$$

- See next figure



What is linearization?

Very, very close to the point $(1, 1)$, the original curve and the tangent line are almost identical.



Linearization around the steady state

The original non-linear model ***The linearized model***

$$c_t^{-\sigma} = \beta(c_{t+1}^{-\sigma} r_{t+1}) \Leftrightarrow \quad \hat{c}_t \approx \hat{c}_{t+1} - \frac{1}{\sigma}\hat{r}_{t+1} \quad \text{(L1)}$$

$$y_t/\ell_t = [\theta/(1-\alpha)]c_t^{\sigma} \Leftrightarrow \quad \hat{\ell}_t \approx \hat{y}_t - \sigma \hat{c}_t \quad \text{(L2)}$$

$$k_{t+1} = (1-\delta)k_t + i_t \Leftrightarrow \quad \hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{\imath}_t \frac{\bar{i}}{\bar{k}} \quad \text{(L3)}$$

$$y_t = a_t k_t^{\alpha} \ell_t^{1-\alpha} \Leftrightarrow \hat{y}_t \approx \hat{a}_t + \alpha \hat{k}_t + (1-\alpha)\hat{\ell}_t \quad \text{(L4)}$$

$$c_t + i_t = y_t \Leftrightarrow \quad \hat{y}_t = \hat{c}_t \frac{\bar{c}}{\bar{y}} + \hat{\imath}_t \frac{\bar{i}}{\bar{y}} \quad \text{(L5)}$$

$$r_{t+1} \equiv \alpha(y_{t+1}/k_{t+1}) + 1 - \delta \Leftrightarrow \quad \hat{r}_{t+1} = \left(\frac{\alpha}{\bar{r}} \cdot \frac{\bar{y}}{\bar{k}}\right)\hat{z}_{t+1} \quad \text{(L6)}$$

$$\ln a_t = (1-\rho)\ln \bar{a} + \rho \ln a_{t-1} + \varepsilon_t \Leftrightarrow \quad \hat{a}_t = \rho \hat{a}_{t-1} + \varepsilon_t \quad \text{(L7)}$$

- To simplify, in (L6) we define $\hat{z}_{t+1} \equiv \hat{a}_{t+1} + (\alpha - 1)\hat{k}_{t+1} + (1-\alpha)\hat{\ell}_{t+1}$
- The red ratios have to be substituted next: $\bar{\imath}/\bar{k}$, $\bar{\imath}/\bar{y}$, $\bar{c}/\bar{y}$, $\bar{y}/\bar{k}$, $\alpha/\bar{r}$.

Linearization + uncertainty

1. Substitute the ratios $\left\{\bar{\imath}/\bar{k}, \bar{\imath}/\bar{y}, \bar{c}/\bar{y}, \bar{y}/\bar{k}, \alpha/\bar{r}\right\}$ with their values.
2. Apply the expectations operator $\mathbb{E}_t$ to $(\hat{c}_{t+1}, \hat{r}_{t+1}, \hat{\ell}_{t+1})$. Why only those?

$$\hat{c}_t \approx \hat{c}_{t+1} - \frac{1}{\sigma}\hat{r}_{t+1} \Leftrightarrow \quad \hat{c}_t \approx \mathbb{E}_t\hat{c}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{r}_{t+1} \quad \text{(U1)}$$

$$\hat{\ell}_t \approx \hat{y}_t - \sigma\hat{c}_t \Leftrightarrow \quad \hat{\ell}_t \approx \hat{y}_t - \sigma\hat{c}_t \quad \text{(U2)}$$

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{\imath}_t\frac{\bar{\imath}}{\bar{k}} \Leftrightarrow \quad \hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{\imath}_t \cdot \delta \quad \text{(U3)}$$

$$\hat{y}_t \approx \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{\ell}_t \Leftrightarrow \hat{y}_t \approx \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{\ell}_t \quad \text{(U4)}$$

$$\hat{y}_t = \hat{c}_t\frac{\bar{c}}{\bar{y}} + \hat{\imath}_t\frac{\bar{\imath}}{\bar{y}} \Leftrightarrow \quad \hat{y}_t = \hat{c}_t(1-\phi) + \hat{\imath}_t \cdot \phi \quad \text{(U5)}$$

$$\hat{r}_{t+1} = \left(\frac{\alpha}{\bar{r}} \cdot \frac{\bar{y}}{\bar{k}}\right)\hat{z}_{t+1} \Leftrightarrow \mathbb{E}_t\hat{r}_{t+1} = (1+\beta\delta-\beta)\hat{z}_{t+1} \quad \text{(U6)}$$

$$\hat{a}_t = \rho\hat{a}_{t-1} + \varepsilon_t \Leftrightarrow \quad \hat{a}_t = \rho\hat{a}_{t-1} + \varepsilon_t \quad \text{(U7)}$$

In (U6) we define $\hat{z}_{t+1} \equiv \hat{a}_{t+1} + (\alpha-1)\hat{k}_{t+1} + (1-\alpha)\mathbb{E}_t\hat{\ell}_{t+1}$

Reducing the number of equations

- By inserting (U6) into (U1) we can obtain:

$$\hat{c}_t \approx \mathbb{E}_t\hat{c}_{t+1} - \frac{\varphi}{\sigma}\left[\hat{a}_{t+1} + (\alpha-1)\hat{k}_{t+1} + (1-\alpha)\mathbb{E}_t\hat{\ell}_{t+1}\right], \quad \varphi \equiv 1+\beta\delta-\beta$$

- Equalizing (U4) & (U5), solving for $\hat{\imath}_t$, and inserting this result into (U3), we get:

$$\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t + \frac{\delta}{\phi}\hat{a}_t - \frac{\delta(1-\phi)}{\phi}\hat{c}_t + \frac{\delta(1-\alpha)}{\phi}\hat{\ell}_t \ , \quad \phi \equiv \frac{\alpha\delta}{\beta^{-1}+\delta-1}$$

- We have [***2 equations x 4 variables***]: $\hat{c}_t, \hat{k}_t, \hat{a}_t, \hat{\ell}_t$. We need to bring in two more equations. Insert (U4) into (U2) and we get:

$$\mathbb{E}_t\hat{\ell}_{t+1} \approx \frac{1}{\alpha}\left(\hat{a}_{t+1} + \alpha\hat{k}_{t+1} - \sigma\mathbb{E}_t\hat{c}_{t+1}\right)$$

- The last one is the law of motion for the technology:

$$\hat{a}_{t+1} = \rho\hat{a}_t + \varepsilon_{t+1}$$

The model ready for for the computer

- The 4 equations to be used in the initial simulation: $\hat{a}_t, \hat{k}_t, \hat{c}_t, \hat{\ell}_{t_t}$

Variables	Equations
Technology	$\hat{a}_{t+1} = \rho \cdot \hat{a}_t + \varepsilon_{t+1}$
Capital	$\hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t + \frac{\delta}{\phi}\hat{a}_t - \frac{\delta(1-\phi)}{\phi}\hat{c}_t + \frac{\delta(1-\alpha)}{\phi}\hat{\ell}_t \ , \ \phi \equiv \frac{\alpha\delta}{\beta^{-1}+\delta-1}$
Labor	$\mathbb{E}_t\hat{\ell}_{t+1} \approx \frac{1}{\alpha}\left(\hat{a}_{t+1} + \alpha\hat{k}_{t+1} - \sigma\mathbb{E}_t\hat{c}_{t+1}\right)$
Consumption	$\hat{c}_t \approx \mathbb{E}_t\hat{c}_{t+1} - \frac{\varphi}{\sigma}\left[\hat{a}_{t+1} + (\alpha-1)\hat{k}_{t+1} + (1-\alpha)\mathbb{E}_t\hat{\ell}_{t+1}\right] \ , \ \varphi \equiv 1 + \beta\delta - \beta$

- The remaining 3 variables can be obtained from the following equations:
  - From (U4): $\hat{y}_t \approx \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{\ell}_t$
  - From (U5): $\hat{\imath}_t = \frac{1}{\phi}\left[\hat{y}_t - (1-\phi)\hat{c}_t\right]$
  - From (U6): $\hat{r}_t = \varphi\left[\hat{a}_t + (\alpha-1)\hat{k}_t + (1-\alpha)\hat{\ell}_t\right]$
- 7 endogenous $\left(\hat{a}_t, \hat{k}_t, \hat{c}_t, \hat{\ell}_t, \hat{y}_t, \hat{\imath}_t, \hat{r}_t\right)$; 1 exogenous (ε_t).

5. The Blanchard-Kahn conditions

The Blanchard-Kahn (BK) conditions

- The model can be written in ***state space form*** as:

$$\mathcal{A}\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \mathcal{B}\begin{bmatrix} w_t \\ v_t \end{bmatrix} + \mathcal{C}\begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} + \mathcal{D}$$

  w is a vector of backward-looking variables (it may include static vriables as well), and v is a vector of forward looking variables.

- Multiplying both sides of (BK1) by $\mathcal{A}^{-1}$, leads to:

$$\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \underbrace{\mathcal{A}^{-1}\mathcal{B}}_{\mathcal{R}}\begin{bmatrix} w_t \\ v_{t+1} \end{bmatrix} + \underbrace{\mathcal{A}^{-1}\mathcal{C}}_{\mathcal{U}}\begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix} + \underbrace{\mathcal{A}^{-1}\mathcal{D}}_{\mathcal{H}}$$

- The BK conditions: for a model to have a unique and stable solution, the matrix $\mathcal{R}$ has (m, n) eigenvalues such that $|m_v| > 1$ and $|n_w| < 1$.

The Jordan decomposition

- The ***Jordan decomposition*** was the algebra technique used by Blanchard-Kahn (1980) to solve a DSGE model.
- Suppose we have a square matrix $\mathcal{R}$
- The Jordan decomposition of $\mathcal{R}$ is given by:

$$\mathcal{R} = P\Lambda P^{-1}$$

- P contains as columns the ***eigenvectors*** of $\mathcal{R}$
- Λ is a diagonal matrix containing the ***eigenvalues*** of $\mathcal{R}$ in the main diagonal.
- P^{-1} is the inverse of P

Blanchard, O. J., & Kahn, C. M. (1980). The Solution of Linear Difference Models under Rational Expectations. *Econometrica*, 48(5), 1305–1311.

Apply the Jordan decomposition to the model

- Our system was given by:

$$\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \mathcal{R} \begin{bmatrix} w_t \\ v_t \end{bmatrix} + \mathcal{U} \begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix}$$

- Apply the **Jordan decomposition** $\mathcal{R} = P\Lambda P^{-1}$ to (BK3), and we get:

$$\begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = P\Lambda P^{-1} \begin{bmatrix} w_t \\ v_t \end{bmatrix} + \mathcal{U} \begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix}$$

- Multiply both sides of (BK4) by P^{-1} :

$$P^{-1} \begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix} = \Lambda P^{-1} \begin{bmatrix} w_t \\ v_t \end{bmatrix} + \underbrace{P^{-1}\mathcal{U}}_{\mathcal{M}} \cdot \begin{bmatrix} \varepsilon^w_{t+1} \\ \varepsilon^v_{t+1} \end{bmatrix}$$

- Next we have to apply a ***partition*** of the matrices above: $P^{-1}, \Lambda, \mathcal{M}$.

Matrices partitions

- Assume that there are no shocks affecting the forward-looking block:

$$\varepsilon^v_t = 0, \forall t$$

- Next, apply a partition to the matrices: $P^{-1}, \Lambda, \mathcal{M}$:

$$\underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_{t+1} \\ \mathbb{E}_t v_{t+1} \end{bmatrix}}_{\mathbb{E}_t \begin{bmatrix} \tilde{v}_{t+1} \\ \tilde{v}_{t+1} \end{bmatrix}} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_t \\ v_t \end{bmatrix}}_{\begin{bmatrix} \tilde{w}_t \\ \tilde{v}_t \end{bmatrix}} + \underbrace{\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}}_{M} \begin{bmatrix} \varepsilon^w_{t+1} \\ 0 \end{bmatrix}$$

- Our transformed model looks much easier now:

$$\begin{bmatrix} \tilde{w}_{t+1} \\ \mathbb{E}_t \tilde{v}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{w}_t \\ \tilde{v}_t \end{bmatrix} + \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \cdot \varepsilon^w_{t+1}$$

- Using these partitions, the ***solution*** is given in the next slide

---

Solution to the model

- To solve the model, we have to apply the following analytical solutions, first for the ***non-forward-looking*** block:

$$w^*_{t+1} = \underbrace{[G^{-1}\Lambda_1 G]}_{g} \cdot w^*_t + \underbrace{[G^{-1}M_{11}]}_{h} \cdot \varepsilon_{t+1}$$

- and, then, for the ***forward-looking*** block:

$$v^*_t = \underbrace{[-P_{22}^{-1}P_{21}]}_{f} \cdot w^*_t$$

- with

$$G \equiv P_{11} - P_{12}(P_{22})^{-1}P_{21}$$

- Therefore, the solution requires the ***computation*** of the matrices: g, h, f, G.

6. Solving the model with the computer

Matrices

- The matrices for the model with the endogenous labor supply are:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\alpha} & -1 & 1 & \frac{\sigma}{\alpha} \\ -\frac{\varphi}{\sigma} & -\frac{\varphi(\alpha-1)}{\sigma} & -\frac{\varphi(1-\alpha)}{\sigma} & 1 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \hat{a}_{t+1} \\ \hat{k}_{t+1} \\ \mathbb{E}_t\hat{\ell}_{t+1} \\ \mathbb{E}_t\hat{c}_{t+1} \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} \rho & 0 & 0 & 0 \\ \frac{\delta}{\phi} & \frac{1}{\beta} & \frac{\delta(1-\alpha)}{\phi} & -\frac{\delta(1-\phi)}{\phi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} \hat{a}_t \\ \hat{k}_t \\ \hat{\ell}_t \\ \hat{c}_t \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} \varepsilon^a_{t+1} \\ \varepsilon^k_{t+1} \\ \varepsilon^\ell_{t+1} \\ \varepsilon^c_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathcal{D}}$$

Using the computer to solve the model

- Run the Pluto notebook **RBC_with_Labor.jl**
- The steps are quite simple:
  - Fill in the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$
  - Compute the matrices $\mathcal{R}, \mathcal{U}, \mathcal{H}$
  - Check the BK conditions: the eigenvalues of $\mathcal{R}$ must be such that $|m_v| > 1$ and $|n_w| < 1$.
  - If the BK conditions are violated, stop. Otherwise, proceed to the next steps.
  - Compute the matrices g, h, f, G
  - Compute the solution to the model using a for loop
- Once the model is solved, we can plot the IRF and analyze the main outputs of the model from a statistical perspective.

7. Readings

Reading guidelines

- This is an introduction to the Real Business Cycle model. Therefore, we do not expect students to cover all the fundamental aspects associated with a sophisticated model in modern macroeconomics.
- For example, the linearization of the model can be left for further studies in more advanced courses. Everything else is not very complicated, and students should be able to understand it.
- The study of this model can be understood from three different perspectives: (i) the basic techniques, (ii) technical details that go beyond the basic issues, and (iii) a critical appraisal of theoretical and empirical issues.

Reading sources

- In this course, we concentrate only on the basic issues. Therefore, students should read the lecture notes by *Mendes, V. (2026). The Real Business Cycle Model: A Step-by-Step Solution* here. Students can skip the section about "linearization".
- For those who want to delve deeper into the model and the associated techniques, we do not know of anything better than an old set of notes by *Krueger, D. (2007). Quantitative Macroeconomics: An Introduction, University of Pennsylvania*, here. They are old but may still be the best for this task.
- If students want to get a feeling of major theoretical and empirical controversies about this model (as well as more recent developments), an excellent source of information is the chapter by *Mitman, K. (2025). "Chapter 14: Real Business Cycles"*, in the online/huge textbook "Macroeconomics", by Azzimonti, M. et al. (2025), available here.

Bibliography